

Novel Ansatzes and Scalar Quantities in Gravito-Electromagnetism

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Abstract In this work, we focus on the theory of Gravito-Electromagnetism (GEM) – the theory that describes the dynamics of the gravitational field in terms of quantities met in Electromagnetism – and we propose two novel forms of metric perturbations. The first one is a generalisation of the traditional GEM ansatz, and succeeds in reproducing the whole set of Maxwell’s equations even for a dynamical vector potential \mathbf{A} . The second form, the so-called alternative ansatz, goes beyond that leading to an expression for the Lorentz force that matches the one of Electromagnetism and is free of additional terms even for a dynamical scalar potential Φ . In the context of the linearised theory, we then search for scalar invariant quantities in analogy to Electromagnetism. We define three novel, 3rd-rank gravitational tensors, and demonstrate that the last two can be employed to construct scalar quantities that succeed in giving results very similar to those found in Electromagnetism. Finally, the gauge invariance of the linearised gravitational theory is studied, and shown to lead to the gauge invariance of the GEM fields \mathbf{E} and \mathbf{B} for a general configuration of the arbitrary vector involved in the coordinate transformations.

Keywords General Relativity · Gravito-Electromagnetism · Maxwell Equations · Lorentz Force · Scalar Invariants · Gauge Invariance

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1 Introduction

Of all the forces in nature, the gravitational force has proven to be the most resistant to being incorporated into a common framework that would unify all possible interactions. Whereas the electromagnetic, weak and strong interactions have all been successfully described by gauge field theories, gravity is accurately described by Einstein's General Theory of Relativity, a mathematical theory based on tensors rather than on four-vectors. An intrinsic difference arises regarding the spin of the fundamental degrees of freedom in each case: while all gauge bosons have spin one, gravitons have spin two; this inevitably affects the type of symmetry present in each theory, and eventually determines the mathematical framework that describes it best.

At the same time, gravity could not be more similar to Electromagnetism (EM) at the classical level. The similarity of the equations obeyed by the Newtonian and Coulomb potential was noticed centuries ago, and this analogy was re-inforced after the discovery of the Lense-Thirring effect [1] where the angular momentum of a rotating body may be interpreted at large distances as a gravitational 'magnetic' field. All these cultivated the impression that the unification of gravity with electromagnetism would be straightforward and imminent. Numerous attempts have therefore appeared in the literature over a century-long period including the Kaluza-Klein theory [2], the string and M-theory [3][4], the loop quantum gravity [5], as well as a number of other geometric theories, classical or quantum, that have attempted to connect gravity and electromagnetism [6][7] (for a more extensive list of references including even earlier works, see [8][9]). So far, the construction of a robust mathematical or geometrical theory, in the context of which the unification of all forces could be realised, is still missing.

A different approach, that has attracted considerable attention over the years, is the one adopted in the theory of Gravito-electromagnetism (GEM) [10][11][12][13][14][15]. In this, the dynamics of the gravitational field is described in terms of quantities met in electromagnetism. The mathematical framework of GEM is that of the General Theory of Relativity, and more specifically, the perturbed Einstein's field equations at linear approximation. The gravitational perturbations $h_{\mu\nu}$ may be expressed in terms of a scalar Φ and a vector potential \mathbf{A} , that at large distances are associated to the Newtonian potential and angular momentum of the source. The linearised field equations then reduce to a set of equations with a close similarity to Maxwell's equations in electromagnetism. The GEM fields \mathbf{E} and \mathbf{B} are defined in terms of the GEM potentials Φ and \mathbf{A} in a way similar to the usual electro-magnetic. In the absence of a true unification scheme, the theory of gravito-electromagnetism has provided a new perspective on the description and understanding of the gravitational field; in addition, it has helped to investigate in depth the analogy between these two fundamental forces, and has guided us to look for gravitational analogues of electromagnetic phenomena in the context of General Relativity.

In a previous work of ours [15], we performed a comprehensive analysis of the set of equations that follow from the linearised field equations for the so-called traditional ansatz for the metric perturbations employed in GEM [12]. Our results demonstrated that the set of equations for the GEM fields reduce to Maxwell's equations but only under the assumption of a static vector potential \mathbf{A} - in fact, the staticity of \mathbf{A} was dictated by the neglected spatial component of the transverse gauge condition, usually adopted in GEM, and was thus an intrinsic feature of the theory. In addition, we showed that the spatial components of the field equations, that are also often ignored in the literature, carry important pieces of information: at times, they may impose unphysical or over-restrictive constraints to the fields or matter distribution of the theory, and therefore should be properly taken into account. Finally, the exact form of the geodesics equation, that in the context of GEM reduces to a form similar to that of the equation for the Lorentz force, was derived and showed to contain additional terms that may not be easily ignored, even in the non-relativistic limit, unless the scalar potential Φ is also static.

An important conclusion that followed from the analysis of [15] was that the form of the gravitational perturbations significantly affects the form of the field equations, the expression for the Lorentz force and the form of any additional constraints. Indeed, an alternative ansatz employed in [15] was shown to have attractive features such as the complete absence of additional constraints or the absence of additional terms in the expression of Lorentz force. However, this was of limited physical relevance as the analysis was restricted only to a vacuum configuration. In the present work, motivated by our previous findings, we introduce two novel forms of metric perturbations. The first one is a generalisation of the traditional ansatz employed in GEM that allows small but non-vanishing spatial components of the metric perturbations $\tilde{h}_{\mu\nu}$. Although the \tilde{h}_{ij} components are not involved in the derivation of Maxwell's equations, due to the tensorial structure of the theory, their absence or presence affects other important equations of the theory. Indeed, we show that the set of Maxwell's equations are exactly reproduced, and the presence of all terms involving time-derivatives of the vector potential \mathbf{A} are now restored.

The second novel form of perturbations combines the attractive features of the first one, the *generalised traditional ansatz*, and the one employed in [15]. This *alternative ansatz* has as its core idea the introduction of the scalar potential Φ into the spatial components \tilde{h}_{ij} , too, of the metric perturbations. Then, a cancellation of potentially harmful terms in the expression of the Lorentz force leaves behind a minimal form identical, at first approximation, to that in Electromagnetism. In addition, no unphysical constraints on matter or field configurations arise in the theory, and the set of Maxwell's equations is again restored for a dynamical vector potential \mathbf{A} .

What is also important in establishing the extent of analogy between gravity and EM is the construction of scalar invariant quantities in the context of GEM similar to those in Electromagnetism. Previous attempts have appeared in the literature before [13] where either the Weyl or the Riemann tensor is employed for this purpose. Here, we search for scalar quantities defined di-

rectly in the context of the linearised gravitational theory of GEM. To this end, we define three novel 3rd-rank gravitational tensors in terms of which we construct scalar quantities, compute their expressions for both ansatzes and evaluate their role as analogues of the scalar quantities of EM.

Finally, we turn our attention to the gauge invariance of the linearised gravitational theory. It is well known that this may be interpreted as a gauge invariance of the GEM fields \mathbf{E} and \mathbf{B} [12]. However, this has been demonstrated for a particular type of coordinate transformations. In the context of the present analysis, we perform a comprehensive analysis, and derive the most general constraints that the coordinate transformation should obey in order for the gauge invariance of the GEM fields to hold.

The outline of our paper is as follows: in Section 2, we present the theoretical framework of our analysis, review the most basic assumptions and equations of GEM, and discuss the weak points of the traditional analysis. In Section 3, we present the two novel forms of gravitational perturbations, and in each case we derive the complete set of field equations, gauge condition and geodesics equation. Then, in Section 4, we focus on the construction of scalar quantities in terms of three novel gravitational tensors, and we compute their expressions for both metric ansatzes. In Section 5, we address the topic of the gauge invariance, derive the full set of constraints on the coordinate transformations and look for the most general configuration. Finally, we present our conclusions in Section 6.

2 The Theoretical Framework

Our theoretical framework will be the one developed in the context of the theory of Gravito-electromagnetism (GEM), therefore we start our analysis by presenting its basic assumptions and equations [11][12]. The general metric tensor is assumed to be written as

$$g_{\mu\nu}(x^\mu) = \eta_{\mu\nu} + h_{\mu\nu}(x^\mu), \quad (2.1)$$

where $\eta_{\mu\nu}$ is the Minkowski metric of the flat spacetime¹ and $h_{\mu\nu}$ are the metric perturbations. The latter are functions of $x^\mu = (ct, \mathbf{x})$ and are associated to the presence of gravitating bodies. They are also assumed to obey the inequality $|h_{\mu\nu}| \ll 1$, and therefore a linear-approximation analysis may be followed for their study.

By following a standard procedure [16][15], the perturbed Einstein's equations take the form

$$G_{\mu\nu} = \frac{1}{2} (h^\alpha_{\mu,\nu\alpha} + h^\alpha_{\nu,\mu\alpha} - \partial^2 h_{\mu\nu} - h_{,\mu\nu} - \eta_{\mu\nu} h^{\alpha\beta}_{,\alpha\beta} + \eta_{\mu\nu} \partial^2 h) = kT_{\mu\nu}, \quad (2.2)$$

¹ Throughout this work, we will use the $(+1, -1, -1, -1)$ signature for the Minkowski tensor $\eta_{\mu\nu}$.

where $k = 8\pi G/c^4$, $\partial^2 \equiv \eta^{\mu\nu} \partial_\mu \partial_\nu$ and $T_{\mu\nu}$ is the energy-momentum tensor. A new form of the metric perturbations, defined through the relation [16]

$$\tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \quad (2.3)$$

results into a simpler form of the field equations, namely

$$\tilde{h}^\alpha_{\mu,\nu\alpha} + \tilde{h}^\alpha_{\nu,\mu\alpha} - \partial^2 \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \tilde{h}^{\alpha\beta}_{,\alpha\beta} = 2k T_{\mu\nu}. \quad (2.4)$$

Finally, the energy-momentum tensor is assumed to be described by the expression $T_{\mu\nu} = \rho u_\mu u_\nu$, where ρ is the mass/charge density in the context of GEM and $u^\mu = (u^0, u^i) = (c, \mathbf{u})$ is the velocity of the source.

In the context of the traditional ansatz adopted in GEM, the components of the metric perturbations $\tilde{h}_{\mu\nu}$ have the form [12]

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}, \quad \tilde{h}_{0i} = \frac{2A_i}{c^2}, \quad \tilde{h}_{ij} = \mathcal{O}(c^{-4}), \quad (2.5)$$

where the scalar $\Phi(x^\mu)$ and vector $\mathbf{A}(x^\mu)$ functions are the so-called gravito-electromagnetic potentials defined in analogy with electromagnetism. In reality, Φ is the Newtonian gravitational potential while \mathbf{A} is associated to the angular-momentum vector, if existent, of the massive body. The spatial components \tilde{h}_{ij} of the perturbations are assumed to be negligible – due to the suppression of the corresponding source by a $1/c^4$ factor – and are thus ignored throughout the analysis. Working in the transverse gauge, i.e. $\tilde{h}^{\mu\nu}_{,\nu} = 0$, whose $(\mu = 0)$ component reduces to the analog of the Lorentz condition

$$\frac{1}{c} \partial_t \Phi + \partial_i \left(\frac{A^i}{2} \right) = 0, \quad (2.6)$$

the (00) and (0i) components of the field equations (2.4) take the form

$$\partial^2 \Phi = -4\pi G \rho, \quad (2.7)$$

and

$$\partial^2 \left(\frac{A^i}{2} \right) = -\frac{4\pi G}{c} j^i, \quad (2.8)$$

respectively, where $j^i \equiv \rho u^i$. By employing the following definitions of the gravito-electromagnetic fields [12] in terms of the GEM potentials

$$\mathbf{E} \equiv -\frac{1}{c} \partial_t \left(\frac{\mathbf{A}}{2} \right) - \nabla \Phi, \quad \mathbf{B} \equiv \nabla \times \left(\frac{\mathbf{A}}{2} \right), \quad (2.9)$$

Eqs. (2.7) and (2.8) adopt the form of two Maxwell-like equations for \mathbf{E} and \mathbf{B} , namely

$$\nabla \cdot \mathbf{E} = 4\pi G \rho, \quad \nabla \times \mathbf{B} = \frac{1}{c} \partial_t \mathbf{E} + \frac{4\pi G}{c} \mathbf{j}. \quad (2.10)$$

The supplementary assumption was also made that the vector potential be static, $\partial_t A^i = 0$. The definitions of the fields Eqs. (2.9) may in turn take the form of the remaining two Maxwell equations

$$\nabla \times \mathbf{E} = \frac{1}{c} \partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \quad (2.11)$$

Finally, the spatial components of the geodesics equation

$$\frac{d^2 x^\rho}{ds^2} + \Gamma_{\mu\nu}^\rho \frac{dx^\mu}{ds} \frac{dx^\nu}{ds} = 0, \quad (2.12)$$

in the same linear approximation, and in the non-relativistic limit where $ds^2 \simeq c^2 dt^2$, may be collectively written as [12]

$$\ddot{x}^i = E^i + \frac{2}{c} F^{ij} u_j, \quad (2.13)$$

where $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. The above has the form of the Lorentz equation of electromagnetism, however, only under the additional assumption that the scalar potential is also time-independent, $\partial_t \Phi = 0$.

In the previous work of ours [15], the above analysis was repeated without the imposition of the gauge condition in order to investigate the reason for the required staticity of the GEM potentials and the role, if any, of the gauge condition in this. Assuming again the ansatz (2.5) for the metric perturbations and neglecting the spatial components \tilde{h}_{ij} , we found that the field equations (2.4) take in fact the following forms:

$$\nabla^2 \Phi = 4\pi G \rho, \quad (2.14)$$

for $(\mu, \nu) = (0, 0)$, and

$$\nabla \left[\nabla \cdot \left(\frac{\mathbf{A}}{2} \right) + \frac{1}{c} \partial_t \Phi \right] - \nabla^2 \left(\frac{\mathbf{A}}{2} \right) = \frac{4\pi G}{c} \rho \mathbf{u}, \quad (2.15)$$

for $(\mu, \nu) = (0, i)$. Equation (2.14), the analogue of Poisson's law, together with Eq. (2.15) take indeed the form of the Maxwell's equations (2.10) but only under the assumption of time-independence of the vector GEM potential, since the anticipated terms involving $\partial_t A^i$ are missing. It is therefore the need for the recovery of the analogy between GEM and electromagnetism that demands the staticity of \mathbf{A} in [12] and not the imposition *per sé* of the gauge condition. Nevertheless, there is an underlying connection since the forgotten $(\mu = i)$ -components of the gauge condition $h^{\mu\nu}_{, \nu} = 0$ in fact reduce to the constraint

$$\frac{1}{c} \partial_t \mathbf{A} = 0, \quad (2.16)$$

that consistently complements the aforementioned requirement of a static vector potential (the role of this constraint in the context of GEM was also studied in [10]f, [11]a,g and [13]).

In addition, let us note that, due to the tensorial structure of gravity, the field equations (2.4) also yield additional constraints arising from the $(\mu, \nu) = (i, j)$ components, namely [15]

$$-\frac{1}{2c^3} \partial_t (\partial_i A^j + \partial_j A^i) + \delta_{ij} \left[\frac{1}{c^4} \partial_t^2 \Phi + \frac{1}{c^3} \partial_t (\partial_k A^k) \right] = \frac{k}{2} \rho u_i u_j. \quad (2.17)$$

The above equations, except in a few analyses [13], were largely ignored in the literature. For $i = j$, they reduce to the relation

$$\partial_t^2 \Phi = -\frac{\pi}{3} \rho |\mathbf{u}|^2, \quad (2.18)$$

with $(u^1)^2 = (u^2)^2 = (u^3)^2$, that demands an isotropic distribution of sources and restricts the magnitude of $\partial_t^2 \Phi$; the off-diagonal components (for $i \neq j$) on the other hand give

$$\partial_0 (\partial_i A^j + \partial_j A^i) = 8\pi G \rho \frac{u_i u_j}{c^2}, \quad (2.19)$$

and dictate that, for a time-independent vector potential as demanded above, the source should be static ($u^i = 0$) or its motion one-directional ($u^i u^j = 0$).

By repeating finally in [15] the derivation of the Lorentz equation from the geodesics equation (2.12), we found that its complete form is given by the expression

$$m \mathbf{a} = \mathbf{F} = m \mathbf{E} \left(1 + \frac{|\mathbf{u}|^2}{c^2} \right) + \frac{4m}{c} \mathbf{u} \times \mathbf{B} + 2m \left[\frac{\mathbf{u}}{c} \frac{\partial_t \Phi}{c} - \frac{\mathbf{u}}{c} \left(\frac{\mathbf{u}}{c} \cdot \mathbf{E} \right) \right], \quad (2.20)$$

where, in accordance to the aforementioned discussion, we have already set $\partial_t \mathbf{A} = 0$. Still, the above expression is an extended one compared to Eq. (2.13) that appeared in [12]. The two additional terms proportional to u^2/c^2 may be indeed safely ignored in the non-relativistic limit. However, the additional term proportional to the combination $\mathbf{u} \partial_t \Phi/c^2$ is not equally suppressed and thus should be taken into account – unless the scalar potential is also time-independent, as assumed in [12]. But although, in this case, this last term that spoils the analogy with the electromagnetism in the expression of the Lorentz force indeed drops out, a new problem arises: through Eq. (2.18), we are forced to restrict our analysis only either to static distributions of matter ($u = 0$), or in pure vacuum ($\rho = 0$); both requirements considerably restrict the physical importance of the achieved analogy with electromagnetism (for a more extended analysis on the link between the time-independence of the GEM potentials and the analogy between gravity and Electromagnetism, see [13]).

3 Novel Ansatzes for the metric perturbations

In the previous section, we have demonstrated that, in the context of the traditional GEM ansatz for the metric perturbations, a static vector potential \mathbf{A} is necessary for the field equations to reduce to a set of Maxwell-like equations, and that only for a static scalar potential Φ the form of the Lorentz force is restored in the non-relativistic limit. In this section, we will consider two alternative, more generalised ansatzes for the metric perturbations, and investigate whether these two problems can be simultaneously resolved without the imposition of the time-independence of the GEM fields.

The metric perturbations $\tilde{h}_{\mu\nu}$ will be assumed to have a form similar to the one given in Eq. (2.5) but with the spatial components \tilde{h}_{ij} not being necessarily negligible. To justify this, let us first derive the constraints that follow from the transverse gauge condition $\tilde{h}^{\mu\nu}_{,\nu} = 0$ while taking into account the presence of the \tilde{h}_{ij} components: for $\mu = 0$, we recover again the usual Lorentz condition (2.6), while for $\mu = i$, we now obtain the constraint

$$\frac{2}{c^3} \partial_t A^i + \partial_j \tilde{h}^{ij} = 0. \quad (3.1)$$

We therefore conclude that the time-dependence of the vector GEM potential \mathbf{A} is directly related to the spatial components of the metric perturbations \tilde{h}_{ij} . Thus, in case these components are altogether ignored in the analysis, as in the traditional GEM ansatz [12], the time-dependence of the vector potential is automatically eliminated.

In the next two subsections, we will therefore assume that the spatial components of the metric perturbations \tilde{h}_{ij} can indeed adopt small but non-negligible values. In the first case, \tilde{h}_{ij} will again be suppressed by a $1/c^4$ factor, as in Eq. (2.5), but an explicit form for it will be used in the analysis in order to investigate its full effect; in the second case, a more educated form of the metric perturbations will be introduced in which the \tilde{h}_{ij} components will consist of a suppressed $\mathcal{O}(1/c^4)$ term and a more dominant $\mathcal{O}(1/c^2)$ term.

3.1 A generalised form of the traditional ansatz

Here, we extend the traditional form assumed for the metric perturbations in the context of GEM [12], and write the following generalised form

$$\tilde{h}_{00} = \frac{4\Phi}{c^2}, \quad \tilde{h}_{0i} = \frac{2A_i}{c^2}, \quad \tilde{h}_{ij} = \frac{2\lambda}{c^4} \eta_{ij} + \frac{2}{c^4} d_{ij}. \quad (3.2)$$

In accordance to the aforementioned discussion, the spatial components \tilde{h}_{ij} are assumed to be small but nevertheless non-vanishing; we expect that this will restore the time-dependence of the vector potential \mathbf{A} in the analysis. In addition, their expression may be decomposed, in the most general case, into two parts: one involving a scalar function λ and one that is proportional to

a second-rank symmetric tensor d_{ij} – the latter will be taken to be traceless since this choice simplifies significantly the subsequent analysis and results.

Let us first address the question of the form of the field equations. Considering the (00) component of the perturbed Einstein's equations (2.4), we derive the equation

$$-\frac{4}{c^2} \eta^{ij} \partial_i \partial_j \Phi - \partial^k \partial^l \tilde{h}_{kl} = 2k T_{00}. \quad (3.3)$$

Employing the gauge condition (3.1), the above may be rewritten as

$$-\delta^{ij} \partial_i \partial_j \Phi - \frac{1}{2c} \partial_i (\partial_t A^i) = -4\pi G \rho. \quad (3.4)$$

Defining the GEM field \mathbf{E} as in Eq. (2.9), the above adopts a form identical to the first Maxwell-like equation (2.10). Note that, as expected, the presence of the \tilde{h}_{ij} component has restored the term proportional to $(\partial_t A^i)$ in the above equation through the improved gauge condition (3.1). Moving to the (0i) component of Einstein's equations, this is found to be

$$\frac{4}{c^3} \partial_t \partial_i \Phi + \frac{2}{c^2} \partial_i (\partial_j A^j) - \frac{2}{c^2} \eta^{kj} \partial_k \partial_j A_i + \frac{1}{c} \partial_t \partial^k \tilde{h}_{ki} = 2k T_{0i}. \quad (3.5)$$

Again, using the additional gauge condition (3.1), we obtain

$$\frac{1}{2} [\partial_i (\partial_j A^j) - \delta^{kj} \partial_k \partial_j A^i] - \frac{1}{c} \partial_t \left(-\partial_i \Phi - \frac{1}{2c} \partial_t A^i \right) = -\frac{4\pi G}{c} \rho u^i. \quad (3.6)$$

If we use the definition of the GEM field \mathbf{B} , as this is given in Eq. (2.9), the above equation again reduces to the second Maxwell-like equation (2.10) with the $\partial_t A^i$ term present as expected. The remaining two Maxwell-like equations follow without a problem.

Finally, the spatial components (ij) of the Einstein's equations (2.4) lead to the additional relation

$$\begin{aligned} & \frac{2}{c^3} \partial_t (\partial_i A_j + \partial_j A_i) + \partial^k \partial_i \tilde{h}_{kj} + \partial^k \partial_j \tilde{h}_{ki} - \frac{1}{c^2} \partial_t^2 \tilde{h}_{ij} - \eta^{kl} \partial_k \partial_l \tilde{h}_{ij} \\ & - \eta_{ij} \left(\frac{4}{c^4} \partial_t^2 \Phi + \frac{4}{c^3} \partial_t \partial_k A^k + \partial^k \partial^l \tilde{h}_{kl} \right) = 2k T_{ij}. \end{aligned} \quad (3.7)$$

Applying both gauge conditions (2.6) and (3.1), the above simplifies to

$$\partial^2 \tilde{h}^{ij} = \frac{2}{c^4} \partial^2 (\lambda \eta^{ij} + d^{ij}) = -\frac{16\pi G}{c^4} j^i u^j, \quad (3.8)$$

As a result, in this case, the additional relation that follows from the perturbed Einstein's equations – instead of imposing over-restrictive constraints to the fields or charge distribution, as in the traditional GEM ansatz [see Eqs. (2.18)-(2.19)] – simply relates the scalar λ and tensor d_{ij} potentials to the distribution of matter/charge of the system.

We now proceed to the expression of the Lorentz force. The geodesics equation (2.12) involves the Christoffel symbols which, in the linear approximation, assume the form

$$\Gamma_{\mu\nu}^{\alpha} = \frac{1}{2} \eta^{\alpha\rho} (h_{\mu\rho,\nu} + h_{\nu\rho,\mu} - h_{\mu\nu,\rho}). \quad (3.9)$$

We therefore need the original perturbations $h_{\mu\nu}$: by contracting Eq. (2.3) by $\eta^{\mu\nu}$, one finds that $h = -\tilde{h}$; then, the inverse relation between the original and the new perturbations may be written as

$$h_{\mu\nu} = \tilde{h}_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} \tilde{h}. \quad (3.10)$$

For the perturbations considered here, given by Eq. (3.2), we find that

$$\tilde{h} = \frac{4\Phi}{c^2} + \frac{6\lambda}{c^4} + \frac{\eta^{ij}d_{ij}}{c^4} = \frac{4\Phi}{c^2} + \frac{6\lambda}{c^4}, \quad (3.11)$$

where we have used the fact that $\eta^{ij}d_{ij} = 0$. The above, together with Eq. (3.10), leads to the original perturbations $h_{\mu\nu}$, or equivalently to the following spacetime line-element through Eq. (2.1):

$$ds^2 = c^2 \left(1 + \frac{2\Phi}{c^2} - \frac{3\lambda}{c^4} \right) dt^2 - \frac{4}{c} (\mathbf{A} \cdot d\mathbf{x}) dt - \left(1 - \frac{2\Phi}{c^2} - \frac{\lambda}{c^4} \right) \delta_{ij} dx^i dx^j + \frac{2d_{ij}}{c^4} dx^i dx^j. \quad (3.12)$$

The spatial components of the geodesics equation (2.12) have the explicit form

$$\frac{d^2 x^i}{dt^2} + c^2 \Gamma_{00}^i + 2c \Gamma_{0j}^i \frac{dx^j}{dt} + \Gamma_{kj}^i \frac{dx^k}{dt} \frac{dx^j}{dt} = 0. \quad (3.13)$$

Employing the expression of the Christoffel symbols (3.9) and of the initial perturbations $h_{\mu\nu}$ from the line-element (3.12), we easily find the components of $\Gamma_{\mu\nu}^{\alpha}$ - these are listed in Eq. (A.1) of the Appendix. Substituting into Eq. (3.13) and after performing a little bit of algebra, we obtain

$$\begin{aligned} \frac{d^2 x^i}{dt^2} &= E^i + \frac{2}{c} F^{ij} u_j + \frac{1}{c^2} (2u^i \partial_t \Phi + 2u^i u^j \partial_j \Phi - u^2 \partial^i \Phi) \\ &+ \frac{3}{2c^2} \partial_j d^{ij} + \frac{1}{2c^4} (2u^i \partial_t \lambda + 2u^i u^j \partial_j \lambda - u^2 \partial^i \lambda) \\ &- \frac{1}{c^4} (2u^j \partial_t d_j^i + 2u^k u^j \partial_j d_k^i - u^k u^j \partial^i d_{jk}). \end{aligned} \quad (3.14)$$

Note that above we have used the definition $F_{ij} \equiv \partial_i A_j - \partial_j A_i$. The above expression provides a generalised form for the Lorentz force in the context of GEM. By setting $d_{ij} = 0$ and $\lambda = 0$, we recover Eq. (2.20) for the traditional ansatz (2.5) with the $u^i \partial_t \Phi / c^2$ term having a significant contribution, even in the non-relativistic limit. In the presence of the d_{ij} and λ potentials, additional

terms arise in the expression of the Lorentz force: although most come with a coefficient of $\mathcal{O}(1/c^4)$ and are thus significantly suppressed, the term $\partial_j d^{ij}/c^2$ cannot again be easily ignored even in the non-relativistic limit.

We thus conclude that the generalised form of the metric perturbations (3.2) employed in this subsection, although it exactly recovers all Maxwell-like equations without demanding the staticity of the vector potential \mathbf{A} , it cannot avoid the presence of corrections in the expression of the Lorentz force. Before, however, hastening to reject this ansatz, we note the similarity with which the scalar potentials Φ and λ appear in the expression of the Christoffel symbols and of the Lorentz force. In the next subsection, we will therefore investigate whether a more elaborate form of the metric perturbations $\tilde{h}_{\mu\nu}$, in which the Φ and λ potential are related, can ameliorate the problems in the expression of the Lorentz force.

3.2 An alternative form of the metric perturbations

A form of the metric perturbations, in which (00) and (ij) components of $\tilde{h}_{\mu\nu}$ are related, was employed also in our earlier work [15]. In there, the following relation was assumed to hold between the aforementioned components of the metric perturbations

$$\tilde{h}_{ij} = -\tilde{h}_{00} \eta_{ij} . \quad (3.15)$$

Although traditionally it is only the \tilde{h}_{00} component that is associated with the scalar potential Φ , the above assumption introduced a dependence on Φ also in the spatial components of the metric perturbations. The above relation was shown to lead to the cancellation of all the additional terms appearing in the expression of the Lorentz force leaving behind only the well-known form of electromagnetism with the exact same coefficients. The weak point of this ansatz was the fact that it was valid only in vacuum and, although it could successfully describe the dynamics of the fields as they propagate, it failed to provide a robust framework for the study of the fields close to sources.

Guided by the above results, in the present analysis we will consider an alternative form of the metric perturbations, namely

$$\tilde{h}_{00} = \frac{\Phi}{c^2}, \quad \tilde{h}_{0i} = \frac{A_i}{c^2}, \quad \tilde{h}_{ij} = -\gamma \frac{\Phi}{c^2} \eta_{ij} + \frac{1}{c^4} d_{ij} . \quad (3.16)$$

The (00) and (0i) components of $\tilde{h}_{\mu\nu}$, modulo numerical coefficients, are identical to the ones in the previously considered ansatz (3.2). The (ij) component involves again the symmetric, traceless, second-rank tensor d_{ij} suppressed by a factor of $1/c^4$, as well as the scalar potential Φ suppressed only by a factor of $1/c^2$ - the latter addition is necessary if the desired cancellation of terms arising from the \tilde{h}_{00} and \tilde{h}_{ij} components is to be realised. The constant coefficient γ , multiplying Φ in the \tilde{h}_{ij} component, will be determined by demanding the absence of corrections in the expression of the Lorentz force.

To this end, we need again the original perturbations $h_{\mu\nu}$ that appear in the Christoffel symbols (3.9). Employing Eq. (3.16), we find that

$$\tilde{h} = (1 - 3\gamma) \frac{\Phi}{c^2}. \quad (3.17)$$

By using again Eq. (3.10), the spacetime line-element involving the original perturbations $h_{\mu\nu}$ now takes the form

$$ds^2 = c^2 \left[1 + (1 + 3\gamma) \frac{\Phi}{2c^2} \right] dt^2 - \frac{2}{c} (\mathbf{A} \cdot d\mathbf{x}) dt - \left[1 - (1 - \gamma) \frac{\Phi}{2c^2} \right] \delta_{ij} dx^i dx^j + \frac{d_{ij}}{c^4} dx^i dx^j. \quad (3.18)$$

The above leads to the components of the Christoffel symbols (3.9) which are now listed in Eq. (A.2) of the Appendix. Substituting again these components into Eq. (3.13), we find the following result for the Lorentz force

$$\begin{aligned} \frac{d^2 x^i}{dt^2} = & -\frac{(1 + 3\gamma)}{4} \partial^i \Phi - \frac{1}{c} \partial_t A^i + \frac{1}{c} F^{ij} u_j \\ & + \frac{1}{2c^2} (1 - \gamma) (u^i \partial_t \Phi + u^i u^j \partial_j \Phi - u^2 \partial^i \Phi) \\ & - \frac{1}{2c^4} (2u^j \partial_t d_{ij}^i + 2u^k u^j \partial_j d_{ik}^i - u^k u^j \partial^i d_{jk}) . \end{aligned} \quad (3.19)$$

For $\gamma = 0$, the analysis reduces again to that of the traditional ansatz (2.5); however, for $\gamma = 1$, all terms of the order $\mathcal{O}(1/c^2)$ are eliminated leaving behind the minimal expression

$$m \mathbf{a} = \mathbf{F} = m \mathbf{E} + \frac{m}{c} \mathbf{u} \times \mathbf{B} - \frac{m u^j}{2c^4} (2\partial_t d_{ij}^j + 2u^k \partial_j d_{ik}^j - u^k \partial^j d_{jk}), \quad (3.20)$$

under the usual definitions for the GEM fields

$$\mathbf{E} \equiv -\frac{1}{c} \partial_t \mathbf{A} - \nabla \Phi, \quad \mathbf{B} \equiv \nabla \times \mathbf{A}. \quad (3.21)$$

The only additional terms in the expression of the Lorentz force are of $\mathcal{O}(1/c^4)$ that can be safely ignored even for large velocities. In this respect, the result matches the one produced in [15] where the d_{ij} term in the metric perturbations was altogether ignored – as we will see, the inclusion of this term in the context of the present analysis, although of small magnitude, will ensure again the consistency of the set of field equations and their validity even for non-vacuum configurations.

Next, we turn to the constraints that follow from the transverse gauge condition, $\tilde{h}^{\mu\nu}_{;\nu} = 0$. The temporal component $\mu = 0$ gives the exact Lorentz condition of electromagnetism, namely:

$$\frac{1}{c} \partial_t \Phi + \partial_i A^i = 0. \quad (3.22)$$

On the other hand, the spatial component $\mu = i$ gives the following equation

$$\frac{1}{c^3} \partial_t A^i + \partial_j \tilde{h}^{ij} = 0, \quad (3.23)$$

or, equivalently, the relation

$$-\frac{1}{c} \partial_t A^i - \partial_i \Phi = E^i = \frac{1}{c^2} \partial_j d^{ij}, \quad (3.24)$$

that relates the gravitoelectric field with the divergence of the tensor potential d_{ij} . Note that, in the absence of the d_{ij} term, the above condition would demand the vanishing of the GEM field \mathbf{E} and, through the field equations, the vanishing of the source – that was the problem encountered in the analysis of [15] that restricted the validity of the results only in vacuum configurations or far away from the source. Also, we should stress that the Φ term on the left-hand-side of Eq. (3.24) comes, not from the \tilde{h}_{00} component, but from the \tilde{h}_{ij} component in the ansatz (3.2) with $\gamma = 1$. Had this term been absent, it would have been the time-derivative of the vector field associated with the spatial derivative of d_{ij} as in Eq. (3.1); for vanishing d_{ij} , we would have to work again with static fields. Here, as also in the previous subsection, the presence of the d_{ij} potential changes the extra component of the gauge condition from an unphysical constraint to a supplementary equation that relates its spatial derivative to the GEM electric field.

Let us finally address the question of the form of the field equations. These follow quite easily from the ones derived in the previous subsection by performing the changes $\Phi \rightarrow \Phi/4$ and $A^i \rightarrow A^i/2$. Thus, the (00) component of the perturbed Einstein's equations (2.4) takes the form

$$-\delta^{ij} \partial_i \partial_j \Phi - \frac{1}{c} \partial_t (\partial_i A^i) = \partial_i E^i = -16\pi G \rho, \quad (3.25)$$

or

$$\partial^2 \Phi = -16\pi G \rho, \quad (3.26)$$

by the use of both gauge conditions (3.22)-(3.23). Similarly, the (0*i*) component of Einstein's equations is found to have the form

$$\frac{1}{c^2} \partial_t^2 A^i - \delta^{kj} \partial_k \partial_j A^i + \partial_i \left(\frac{1}{c} \partial_t \Phi + \partial_j A^j \right) = -\frac{16\pi G}{c} \rho u^i, \quad (3.27)$$

or the more familiar one

$$\partial^2 A^i = -\frac{16\pi G}{c} j^i, \quad (3.28)$$

again by using Eqs. (3.22)-(3.23). We thus observe that these two components of the field equations adopt indeed forms identical to the Maxwell equations (2.10) apart from a factor of 4 on their right-hand-sides.

Finally, applying both gauge conditions (3.22) and (3.23) to the spatial components (*ij*) of the Einstein's equations, we obtain the simple relation

$$\partial^2 \tilde{h}^{ij} = -\frac{16\pi G}{c^4} j^i u^j, \quad (3.29)$$

Employing the exact form of the \tilde{h}_{ij} perturbations and Eq. (3.26), we obtain the final form

$$\partial^2 d^{ij} = -8\pi G (j^i u^j + \rho c^2 \eta^{ij}). \quad (3.30)$$

Also in this case, the additional relation that follows from the perturbed Einstein's equations simply relates the tensor potential d_{ij} to the distribution of matter/charge of the system.

The above results constitute a significant improvement compared to the ones that follow in the case of the traditional GEM ansatz (2.5). Here, the time-dependence of both GEM potentials is restored: a set of four Maxwell-like equations are recovered not only for static but also for a dynamical vector potential \mathbf{A} , and the expression of the Lorentz force matches the exact electromagnetic one without having to assume that the scalar potential Φ is static, too. Also, as noted above, the field configurations are free from unphysical constraints since the additional components serve to determine the tensor potential d_{ij} whose contribution to observable effects, such as the Lorentz force, remains always suppressed.

The aforementioned analysis improves also the one presented in [15]. Due to the presence of the d_{ij} tensor potential, no unphysical constraints, such as the vanishing of the source terms, arise here. The derived equations describe the fields not only in vacuum but also in the presence of sources. The only shortcoming that destroys the exact similarity to the corresponding formulae of electromagnetism is the numerical coefficient of 16π , instead of 4π , on the right-hand-side of the Maxwell-like equations. One could suggest the redefinition of the GEM potentials, as these appear in the perturbations ansatz (3.16), according to the rule

$$\Phi \longrightarrow 4\Phi^*, \quad \mathbf{A} \longrightarrow 4\mathbf{A}^*, \quad (3.31)$$

where Φ^* and \mathbf{A}^* are now the true potentials. Then, defining the gravitoelectric and gravitomagnetic field as

$$\mathbf{E} \equiv -\frac{1}{c} \partial_t \mathbf{A}^* - \nabla \Phi^*, \quad \mathbf{B} \equiv \nabla \times \mathbf{A}^*, \quad (3.32)$$

one may easily see that Eqs. (3.26) and (3.28) along with Eqs. (3.32) adopt the correct form, namely

$$\nabla \cdot \mathbf{E} = 4\pi G \rho, \quad \nabla \times \mathbf{B} = \partial_t \mathbf{E} + \frac{4\pi G}{c} \mathbf{j}, \quad (3.33)$$

and

$$\nabla \times \mathbf{E} = \partial_t \mathbf{B}, \quad \nabla \cdot \mathbf{B} = 0. \quad (3.34)$$

However, in this case, the redefinition of the GEM fields would unavoidably introduce an additional numerical factor of 4 in the expression of the Lorentz force (3.20) that would again destroy the exact similarity with electromagnetism. We may thus conclude that although the perturbations ansatz employed here has come a long way in overcoming major obstacles in the analogy

between gravity and electromagnetism, an exact matching between the corresponding field equations is still eluding us. An alternative perhaps suggestion would be the redefinition of the matter/charge density according to the rule $\rho_e = 4\rho_m$, where $\rho_{m(e)}$ is the matter and corresponding charge density, respectively: in this case, all the positive features of the perturbations ansatz employed here would be retained, and the exact matching in the form of all equations between gravity and electromagnetism would be achieved.

4 Scalar Invariant Quantities in GEM

Pursuing further the analogy between GEM and electromagnetism, in this section we will search for scalar quantities – and thus invariant under coordinate transformations – constructed in the context of GEM and conveying information similar to that in electromagnetism. As is well-known, in pure electromagnetism one may construct two such invariant quantities: the inner product of the electromagnetic field strength tensor

$$F^{\mu\nu} F_{\mu\nu} = 2(B^2 - E^2), \quad (4.1)$$

and its product with its dual tensor, i.e.

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}, \quad (4.2)$$

namely,

$$\tilde{F}^{\mu\nu} F_{\mu\nu} = 4 \mathbf{E} \cdot \mathbf{B}. \quad (4.3)$$

Both quantities (4.1) and (4.3) are also gauge invariant, with the first one being of fundamental importance as it appears in the expression of both the Lagrangian of the electromagnetic field and its energy-momentum tensor.

Constructing similar quantities in the context of GEM is not an easy task. Although scalar quantities may be easily constructed in terms of gravitational quantities, such as the Riemann and the Ricci tensor, these involve second derivatives of the metric tensor and therefore second derivatives of the GEM potentials Φ and \mathbf{A} . As a result, any invariant quantity constructed in this way would unavoidably involve not the GEM fields \mathbf{E} and \mathbf{B} , as the ones in Eqs. (4.1) and (4.3), but their first derivatives.

It is possible to construct invariants similar to those of electromagnetism by using the Weyl or the Riemann tensor [13], however this approach deviates from the one followed in the traditional formulation of GEM. Here, we will investigate whether it is possible to construct scalar invariants by using also gravitational quantities but in the context of the linearised approach followed so far in this work.

4.1 A generalised field-strength tensor

In electromagnetism, the field-strength tensor $F^{\mu\nu}$ appears also in the relativistic field equations, namely

$$\partial_\mu F^{\mu\nu} = 4\pi J^\nu. \quad (4.4)$$

Motivated by this, we turn for guidance to the perturbed Einstein's equations (2.4). In the context of gravity, these can be alternatively written as [9]

$$G_{\mu\nu} = \frac{1}{2} \partial^\alpha F_{\alpha\mu\nu} = k T_{\mu\nu}, \quad (4.5)$$

where the tensor $F_{\alpha\mu\nu}$ is defined as

$$F_{\alpha\mu\nu} = \partial_\mu \tilde{h}_{\alpha\nu} + \partial_\nu \tilde{h}_{\alpha\mu} - \partial_\alpha \tilde{h}_{\mu\nu} - \eta_{\mu\nu} \partial^\beta \tilde{h}_{\alpha\beta}. \quad (4.6)$$

This new tensor is a third-rank tensor that is also symmetric in the last two indices, therefore it looks distinctly different from the electromagnetic field strength tensor $F_{\mu\nu}$. However, the similarity between Eqs. (4.4) and (4.5) motivates us to consider the following scalar combination

$$F = F^{\alpha\mu\nu} F_{\alpha\mu\nu} \quad (4.7)$$

as a plausible analogue of the electromagnetic invariant (4.1). In what follows, we will compute the aforementioned scalar quantity for the two ansatzes (3.2) and (3.16) for the metric perturbations.

We start with the generalised form (3.2) of the metric perturbations, that succeeded in restoring the time dependence of the vector potential, and we compute the components of the new tensor $F_{\alpha\mu\nu}$ (4.6). Their explicit form is given in Eq. (B.1) in the Appendix. The scalar quantity $F = F^{\alpha\mu\nu} F_{\alpha\mu\nu}$ can be explicitly written in the following way

$$F = F^{000} F_{000} + 2F^{00i} F_{00i} + F^{0ij} F_{0ij} + F^{i00} F_{i00} + 2F^{ij0} F_{ij0} + F^{ijk} F_{ijk}. \quad (4.8)$$

Substituting the components of $F_{\alpha\mu\nu}$ from Eq. (B.1) into the above expression, and after some tedious algebra, we find the result ²

$$F = \frac{1}{c^4} \left[64B^2 - 16E^2 - 32(\partial_t \Phi)^2 + 4(\partial_i A_j + \partial_j A_i)^2 + 4(\partial_i A^i)^2 \right. \\ \left. - 16\left(\partial^i A_i\right)\left(\frac{2}{c}\partial_t \Phi + \partial_j A^j\right) + 12\left(\frac{2}{c}\partial_t \Phi + \partial_j A^j\right)^2 \right] + R, \quad (4.9)$$

² From this point onwards, and in order to simplify the analysis, we will set $\lambda = 0$; as is clear by looking at the components (B.1), all terms related to the λ potential will be suppressed by, at least, a factor of $\mathcal{O}(1/c^6)$ in the expression of the F invariant quantity (4.7) and thus will be negligible – by using the components (B.1), the interested reader may compute the full expression of the F invariant and check that indeed the presence of the scalar potential λ does not alter the results in any significant way.

where – we remind the reader – the GEM fields are defined as

$$E^i \equiv -\frac{1}{c} \partial_t \left(\frac{A^i}{2} \right) - \partial_i \Phi, \quad F_{ij} \equiv \partial_i A_j - \partial_j A_i = -2\varepsilon_{ijk} B_k. \quad (4.10)$$

In addition, R stands for the sum of the sub-dominant terms

$$\begin{aligned} R = & \frac{16}{c^6} \partial^i \Phi \partial^k d_{ik} - \frac{16}{c^7} \partial_t d_{ij} (\partial^i A^j) - \frac{4}{c^8} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk})^2 \\ & + \frac{8}{c^8} (\partial_j d^{ij}) (\partial^k d_{ik}) + \frac{12}{c^{10}} (\partial_t d_{ij})^2. \end{aligned} \quad (4.11)$$

We note that none of the two gauge conditions (2.6) and (3.1) have been used in the calculation leading to Eqs. (4.9) and (4.11). However, according to the discussion of Section 3.1, the implementation of the spatial gauge condition (3.1) is necessary for the restoration of the time-dependence of the vector potential and the appearance of all relevant terms in the Maxwell-like equations. The use of this gauge condition relates the $\partial^k d_{ik}$ factor, appearing in the first term in the expression of R , to the time derivative $\partial_t A_i$; this eventually modifies the coefficients in front of several terms in Eq. (4.9). In addition, if we also apply the Lorentz gauge (2.6), the result is simplified since both terms at the second row of Eq. (4.9) vanish. At the end, we obtain

$$F = \frac{4}{c^4} \left[8(2B^2 - E^2) - 4(\partial_i \Phi)^2 + (\partial_i A_j + \partial_j A_i)^2 + (\partial_i A^i)^2 \right] + \tilde{R}, \quad (4.12)$$

where \tilde{R} is identical to the previous expression R but without its first term.

In conclusion, the scalar quantity $F^{\alpha\mu\nu} F_{\alpha\mu\nu}$ has in fact the desired quadratic dependence on the GEM fields \mathbf{E} and \mathbf{B} , however, the exact result is not satisfactory: the numerical coefficients are not the expected ones and, more importantly, additional terms, of equal magnitude compared to E^2 and B^2 , arise in its expression. Finally, we note the presence of the symmetric combination $(\partial_i A_j + \partial_j A_i)$ which is the result of the symmetry in the last two indices of $F_{\alpha\mu\nu}$, a symmetry that is of course absent in the electromagnetic tensor $F_{\mu\nu}$.

In order to investigate how much the result given in Eq. (4.12) depends on the specific ansatz for the metric perturbations, we will now consider the ansatz (3.16) that has led to the most successful expression of the Lorentz force. Using the expressions for $\tilde{h}_{\mu\nu}$ in this case, we find the components of the generalised tensor $F_{\alpha\mu\nu}$ that are given in Eq. (B.3) of the Appendix. Substituting these into Eq. (4.8) for the scalar $F = F^{\alpha\mu\nu} F_{\alpha\mu\nu}$ and applying both gauge conditions (3.22) and (3.24), we obtain the final result

$$F^{\alpha\mu\nu} F_{\alpha\mu\nu} = \frac{1}{c^4} \left[4(B^2 - E^2) - 10(\partial_i \Phi)^2 + (\partial_i A_j + \partial_j A_i)^2 \right] + \tilde{R}, \quad (4.13)$$

where \tilde{R} stands again for the sum of the sub-dominant terms that now has the form

$$\tilde{R} = \frac{6}{c^6} (\partial_t \Phi)^2 - \frac{4}{c^7} \partial_t d_{ij} (\partial^i A^j) - \frac{1}{c^8} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk})^2$$

$$+ \frac{2}{c^{10}} (\partial_t d_{ij})^2. \quad (4.14)$$

The definitions for the GEM fields that were used in Eq. (4.13) are the ones given in Eq. (3.21). We note that the use of the alternative ansatz (3.16) has improved the expression for the scalar invariant F , yielding the desired combination $(B^2 - E^2)$, however the appearance of additional terms of equal magnitude, although more restricted compared to the case of the ansatz (3.2), still can not be avoided.

In the light of the above results, we conclude that the selection of the generalised field-strength tensor $F^{\alpha\mu\nu}$ (4.6) for the construction of invariant quantities in the context of GEM, although well motivated due to the validity of Eq. (4.5), was not a completely successful one. As a result, we will not attempt to construct here a second invariant of the form $\tilde{F}^{\alpha\mu\nu} F_{\alpha\mu\nu}$, where $\tilde{F}^{\alpha\mu\nu}$ is a dual form of $F^{\alpha\mu\nu}$, as the analogue of the electromagnetic invariant quantity (4.3). Rather, in the next subsection, we will present a second set of gravitational invariants that yield more satisfactory results.

4.2 A novel set of gravitational tensors

We will now consider a new set of gravitational quantities: two third-rank tensors $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ defined through the following expressions

$$Q_{\alpha\mu\nu} \equiv 2\Gamma_{\alpha\mu\nu} - \eta_{\alpha\nu} \partial_\mu h, \quad H_{\alpha\mu\nu} \equiv 2\Gamma_{\mu\alpha\nu} - \eta_{\alpha\mu} \partial_\nu h, \quad (4.15)$$

in terms of the fully covariant form of the Christoffel symbols $\Gamma_{\alpha\mu\nu} = \eta_{\alpha\rho} \Gamma_{\mu\nu}^\rho$, that has the simpler form

$$\Gamma_{\alpha\mu\nu} \equiv \frac{1}{2} (h_{\alpha\mu,\nu} + h_{\alpha\nu,\mu} - h_{\mu\nu,\alpha}). \quad (4.16)$$

Note that, although $\Gamma_{\alpha\mu\nu}$ is symmetric in the last two indices, this symmetry is destroyed at the level of the quantities $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ ³.

The physical motivation for the introduction of these two gravitational tensors lies in the fact that the following scalar combination of them

$$\Lambda_1 \equiv -\frac{1}{4} Q^{\alpha\mu\nu} H_{\alpha\mu\nu}, \quad (4.17)$$

can be shown, by a simple substitution of Eqs. (4.15)-(4.16), to give exactly the Lagrangian of the gravitational field in the weak-field approximation, as this is given in [17], namely

$$\mathcal{L} = \frac{1}{2} \left(\frac{1}{2} \partial_\mu h_{\alpha\nu} \partial^\mu h^{\alpha\nu} - \partial_\mu h_{\alpha\nu} \partial^\alpha h^{\mu\nu} + \partial_\mu h^{\mu\nu} \partial_\nu h - \frac{1}{2} \partial_\mu h \partial^\mu h \right). \quad (4.18)$$

³ We should also mention that the quantities $Q_{\alpha\mu\nu}$, $H_{\alpha\mu\nu}$, $F_{\alpha\mu\nu}$ and $\Gamma_{\mu\nu}^\alpha$ – although in general are not tensors – in the linear approximation and especially under Lorentz transformations behave as tensors.

The variation of the above Lagrangian with respect to the metric perturbations $h_{\mu\nu}$ leads to the perturbed Einstein's equations (2.4). Therefore, the sheer analogy between Λ_1 and the Lagrangian of electromagnetism, $\mathcal{L}_{EM} = -F^{\mu\nu}F_{\mu\nu}/4$, makes the aforementioned scalar combination an excellent candidate for one of the scalar invariant quantities in GEM.

In order to compute the scalar invariant quantity Λ_1 , we need first the components of the novel tensors $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ for the different metric ansatzes. We will start with the generalised form of the metric perturbations given in Eq. (3.2): employing the components of the $\Gamma_{\alpha\mu\nu}$ quantities, listed for convenience in Eq. (B.4) of the Appendix, and the trace relation $h = -\tilde{h}$ along with Eq. (3.11), a straightforward calculation⁴ leads to the components of the $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ tensors presented in Eqs. (B.5) and (B.6), respectively, in the Appendix. Substituting these into Eq. (4.17), we finally find the result

$$-\frac{1}{4}Q^{\alpha\mu\nu}H_{\alpha\mu\nu} = \frac{2}{c^4}(4B^2 - E^2) - R, \quad (4.19)$$

where the sum of the sub-dominant terms R has the form

$$\begin{aligned} R = & \frac{6}{c^5}(\partial_t\Phi)(\partial_t A^i) + \frac{1}{c^6}\left[6(\partial_t\Phi)^2 + \frac{1}{2}(\partial_t A^i)(\partial_t A_i)\right] + \frac{4}{c^7}\partial_t d_{ij}(\partial^i A^j) \\ & + \frac{1}{c^8}(\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk})(\partial^k d^{ij} + \partial^i d^{jk} - \partial^j d^{ik}) - \frac{1}{c^{10}}(\partial_t d_{ij})^2. \end{aligned} \quad (4.20)$$

We observe that the scalar invariant (4.19) yields again the desired terms E^2 and B^2 of the GEM fields – the combination differs again from the expected $(B^2 - E^2)$, however, we note that the coefficient of 4 in front of B^2 matches the coefficient appearing in front of \mathbf{B} in the corresponding expression of the Lorentz force (3.14), a feature that may have an underlying significance – in [12] it was argued that this factor is due to the spin of the gravitational field. What is also important is the fact that the result is completely free of any additional terms of order $\mathcal{O}(1/c^4)$: all terms appearing in R are suppressed by at least an additional factor of c – remarkably, for static configurations most of the additional sub-dominant terms trivially vanish.

Let us also calculate the expression of the Λ_1 scalar invariant for the alternative metric ansatz (3.16). Using the corresponding components of $\Gamma_{\alpha\mu\nu}$ presented in Eq. (B.7), and the trace h [easily deduced from Eq. (3.17) with $\gamma = 1$], we find first the components of the $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ tensors; these are given in Eqs. (B.8) and (B.9) of the Appendix. Employing those in Eq. (4.17), we obtain the result

$$-\frac{1}{4}Q^{\alpha\mu\nu}H_{\alpha\mu\nu} = \frac{1}{2c^4}(B^2 - 2E^2) - \frac{1}{4}\tilde{R}, \quad (4.21)$$

after having used again the gauge conditions (3.22) and (3.23), the first one for simplicity, the second as a pre-requisite for the correct form of field equations.

⁴ Since the presence of the scalar potential λ has again no significant effect on the derived results, for simplicity, we set again $\lambda = 0$.

The sum of the sub-dominant terms \tilde{R} now has the form

$$\begin{aligned}\tilde{R} = & \frac{4}{c^6} [(\partial_t \Phi)^2 + (\partial_t A^i)(\partial_t A_i)] + \frac{4}{c^7} \partial_t d_{ij} (\partial^i A^j) \\ & + \frac{1}{c^8} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk}) (\partial^k d^{ij} + \partial^i d^{jk} - \partial^j d^{ik}) - \frac{1}{c^{10}} (\partial_t d_{ij})^2.\end{aligned}\quad (4.22)$$

Once again, the scalar invariant (4.21) comes out to be free of any additional corrections that are of the same magnitude as E^2 and B^2 – in fact, the next-to-leading order term of $\mathcal{O}(1/c^5)$ is now completely missing. Again, for static potentials, most of the sub-dominant corrections vanish leaving behind only an additional term suppressed by a factor of $\mathcal{O}(1/c^8)$. Although the dominant combination of terms has a minimal, correct form, a superfluous factor of 2 in front of E^2 destroys the perfect analogy.

Our next task is to construct a second invariant quantity in the context of GEM, the analogue of $\tilde{F}F$ of Eq. (4.3). Due to the fact that the Q and H tensors are third-rank tensors, there is a variety of ways that one may construct their dual quantities by employing the antisymmetric $\epsilon^{\mu\nu\rho\sigma}$ tensor. Remarkably, almost all combinations lead to a null result for the corresponding scalar invariant quantity. We have succeeded in finding a non-trivial result only for the following definition of the dual form of the $Q_{\alpha\mu\nu}$ tensor

$$\tilde{Q}^{\alpha\mu\nu} = \epsilon^{\alpha\mu\rho\sigma} Q_{\rho\sigma}{}^\nu. \quad (4.23)$$

By using the above, one may easily construct the scalar invariant quantity

$$\Lambda_2 = \tilde{Q}^{\alpha\mu\nu} H_{\alpha\mu\nu} = \epsilon^{\alpha\mu\rho\sigma} Q_{\rho\sigma}{}^\nu H_{\alpha\mu\nu} = Q_{\rho\sigma\nu} \tilde{H}^{\rho\sigma\nu}. \quad (4.24)$$

The last equality follows easily by defining the dual form of the $H_{\alpha\mu\nu}$ tensor in a similar way, namely as $\tilde{H}^{\rho\sigma\nu} = \epsilon^{\rho\sigma\alpha\mu} H_{\alpha\mu}{}^\nu$; therefore, the two scalar combinations $\tilde{Q}H$ and $\tilde{H}Q$ are in fact identical. In addition, two more scalar invariant quantities could be constructed, namely

$$\Lambda_3 = \tilde{Q}^{\alpha\mu\nu} Q_{\alpha\mu\nu}, \quad \Lambda_4 = \tilde{H}^{\alpha\mu\nu} H_{\alpha\mu\nu}. \quad (4.25)$$

In fact, one may easily show all three scalar invariants Λ_2 , Λ_3 and Λ_4 are again identical: using the definition of the dual tensor in each case and the expressions (4.15) of the Q and H tensors, one arrives at the general relation

$$\tilde{Q}^{\alpha\mu\nu} H_{\alpha\mu\nu} = -\tilde{Q}^{\alpha\mu\nu} Q_{\alpha\mu\nu} = -\tilde{H}^{\alpha\mu\nu} H_{\alpha\mu\nu} = 4\epsilon^{\alpha\mu\rho\sigma} \Gamma_{\rho\sigma}{}^\nu \Gamma_{\mu\alpha\nu}. \quad (4.26)$$

The above result eliminates the apparent freedom in the construction of the second invariant quantity, and allows us to choose any of the above combinations as its functional form. In practice, its expression will be calculated by using the components of the fully covariant Christoffel symbols.

We will calculate the above quantity for both metric perturbations ansatzes (3.2) and (3.16). Starting with the generalised traditional ansatz for the metric

perturbations (3.2) and using the corresponding components of $\Gamma_{\alpha\mu\nu}$ from Eq. (B.4), we obtain the following result

$$\frac{1}{16} \tilde{Q}^{\alpha\mu\nu} H_{\alpha\mu\nu} = -\frac{8}{c^4} \mathbf{E} \cdot \mathbf{B} + Y, \quad (4.27)$$

where the quantity Y stands for the sum of the sub-dominant terms, namely

$$Y = -\frac{1}{c^5} \mathbf{B} \cdot \partial_t \mathbf{A} - \frac{1}{c^6} e^{0ilk} \partial_k d_l^j \left(\partial_i A_j + \partial_j A_i - \frac{2}{c^3} \partial_t d_{ij} \right). \quad (4.28)$$

The quantity $\tilde{Q}_{\alpha\mu\nu} H_{\alpha\mu\nu}$ is found to have indeed the desired form being proportional to the internal product of the GEM fields \mathbf{E} and \mathbf{B} , as in Eq. (4.3). The additional terms appearing in Y are suppressed by, at least, a factor of c and are thus sub-dominant - again, for static configurations the only surviving term is of the order of $\mathcal{O}(1/c^6)$. In addition, due to the way the GEM fields combine in this invariant, any superfluous numerical factor appears as an overall multiplicative factor in front of their internal product, and may be easily absorbed into the definition of the scalar quantity, as was indeed done in Eq. (4.27).

Moving finally to the alternative ansatz for the metric perturbations (3.16), and employing the $\Gamma_{\alpha\mu\nu}$ components - displayed in Eq. (B.7) - in the Λ_2 scalar invariant quantity, we find the result

$$\frac{1}{4} \tilde{Q}^{\alpha\mu\nu} H_{\alpha\mu\nu} = -\frac{5}{c^4} \mathbf{E} \cdot \mathbf{B} + Y, \quad (4.29)$$

where the quantity Y now stands for the following combination

$$Y = -\frac{2}{c^5} \mathbf{B} \cdot \partial_t \mathbf{A} - \frac{1}{c^6} e^{0ilk} \partial_l d_k^j \left(\partial_i A_j + \partial_j A_i - \frac{2}{c^3} \partial_t d_{ij} \right). \quad (4.30)$$

Once again, the Λ_2 invariant yields the correct dependence on the GEM fields, while any additional terms are sub-dominant. We observe that the use of the two different metric ansatzes has resulted in very similar results for the dual scalar invariant quantity, namely in Eqs. (4.27)-(4.28) and (4.29)-(4.30); differences appear only in the numerical factors and not in the functional form, a feature that may hint to some kind of universality of this invariant quantity for a class of metric ansatzes in the context of GEM.

5 Coordinate transformation and GEM gauge invariance

Let us consider the following coordinate transformation

$$x^\mu \rightarrow x'^\mu = x^\mu - \epsilon^\mu(x^\rho), \quad (5.1)$$

where ϵ^μ is an arbitrary vector that, in general, depends on all spacetime coordinates. If we want the above coordinate transformation to be a diffeomorphism, then, under the decomposition (2.1) for the metric tensor and the

assumption that ϵ^μ is also a small quantity, we obtain the perturbations transformation [16][17]

$$h'_{\mu\nu} = h_{\mu\nu} + \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu}. \quad (5.2)$$

One may easily verify that the perturbed Einstein's equations (2.2) remain invariant under the above transformation of $h_{\mu\nu}$, a result that is known as the gauge invariance of the linearised gravitational theory⁵.

Using the above transformation relation for $h_{\mu\nu}$, we may compute the transformation rule of the trace $h = \eta^{\mu\nu} h_{\mu\nu}$, that is

$$h' = h + 2\epsilon^\rho{}_{,\rho}. \quad (5.3)$$

Employing the above, we may find the transformation relation for the new perturbations $\tilde{h}_{\mu\nu}$, defined in (2.3), which has the form

$$\tilde{h}'_{\mu\nu} = \tilde{h}_{\mu\nu} + \epsilon_{\mu,\nu} + \epsilon_{\nu,\mu} - \eta_{\mu\nu} \epsilon^\rho{}_{,\rho}. \quad (5.4)$$

Again, we may easily check that under the above, the novel form of the perturbed Einstein's equations (2.4) remains invariant.

As the components of the perturbations $\tilde{h}_{\mu\nu}$ are directly related to the scalar Φ and vector potential \mathbf{A} , any coordinate transformation causes a change in the potentials themselves. In this section, we would like to investigate whether the gauge invariance of the gravitational linearised theory amounts to a gauge invariance of the GEM fields \mathbf{E} and \mathbf{B} . This question has been addressed in the literature before [12] but for a more special choice of the metric perturbations and under some simplifying assumptions. Here, we will employ the more successful - from the point of view of the restoration of the form of both Maxwell's and Lorentz equations - ansatzes (3.2) and (3.16), and attempt to find the most general configuration for the arbitrary vector $\epsilon^\mu(x^\rho)$ for which the gauge invariance of the GEM fields hold.

We will start with the generalised traditional ansatz (3.2). The transformation rule (5.4) for the \tilde{h}_{00} and \tilde{h}_{0i} components lead to the following transformation of the potentials Φ and \mathbf{A}

$$\Phi \rightarrow \Phi' = \Phi + \frac{c^2}{4} (\partial_0 \epsilon^0 - \partial_i \epsilon^i), \quad (5.5)$$

$$A_i \rightarrow A'_i = A_i + \frac{c^2}{2} (\partial_i \epsilon_0 + \partial_0 \epsilon_i). \quad (5.6)$$

The transformation rule of the \tilde{h}_{ij} components contains the changes of both the scalar potential λ and the symmetric tensor d_{ij} . In order to disentangle these two transformation relations, we employ also the transformation of the trace h , given in Eq. (5.3), and the relation $\tilde{h} = -h$. Then, we find

$$\lambda \rightarrow \lambda' = \lambda - \frac{c^4}{2} (\partial_0 \epsilon^0 + \frac{1}{3} \partial_i \epsilon^i), \quad (5.7)$$

$$d_{ij} \rightarrow d'_{ij} = d_{ij} + \frac{c^4}{2} (\partial_i \epsilon_j + \partial_j \epsilon_i - \frac{2}{3} \eta_{ij} \partial_k \epsilon^k). \quad (5.8)$$

⁵ Here, we work at the lowest order and thus the energy-momentum tensor $T_{\mu\nu}$ is taken to be independent of $h_{\mu\nu}$.

Although the GEM fields depend only on Φ and \mathbf{A} , the presence of λ and d_{ij} , along with their changes under the aforementioned transformation rules, is imperative for the coordinate transformation to “close” – we will return to this point later.

Let us now examine how the GEM fields change under the corresponding changes of the potentials. Starting with the gravito-electric field \mathbf{E} , we write ⁶

$$E^i \rightarrow E'^i = -\partial_i \Phi' - \frac{1}{2} \partial_0 A'^i = E^i + \frac{c^2}{4} (\partial_0^2 \epsilon_i + \partial_i \partial_k \epsilon^k). \quad (5.9)$$

Moving in turn to the gravito-magnetic field \mathbf{B} and employing the relation

$$F_{ij} = -2\epsilon_{ijk} B_k \Rightarrow B^k = \frac{1}{2} \epsilon^{kij} \partial_i A_j, \quad (5.10)$$

we obtain

$$B^k \rightarrow B'^k = B^k + \frac{c^2}{4} \epsilon^{kij} \partial_0 \partial_i \epsilon_j. \quad (5.11)$$

In the context of Electromagnetism, the gauge invariance is the invariance of the electric \mathbf{E} and magnetic field \mathbf{B} under the following changes of the potentials

$$\Phi \rightarrow \Phi' = \Phi - \partial_0 \Lambda, \quad A^i \rightarrow A'^i = A^i + \partial_i \Lambda, \quad (5.12)$$

where Λ is an arbitrary scalar function. In the context of GEM, and for the traditional ansatz (2.5), a similar type of gauge invariance was discussed in [12]. To match the functional forms of the equations (5.5) and (5.6) with those of Eq. (5.12), the spatial components of the ϵ^μ vector were altogether ignored and only the temporal component ϵ^0 was kept. Indeed, under the assumption that $\epsilon^i = 0$, one may easily see that the additional terms in Eqs. (5.9) and (5.11) vanish, and the gauge invariance holds.

However, as with the case of the additional components of the Einstein's equations – which are not necessary for the analogy with the EM but are nevertheless present – also here, the presence of the additional components of the vector field ϵ^μ is just another manifestation of the broader structure of the General Theory of Relativity compared to the U(1) gauge field theory of EM. Over-restrictive choices for the “additional” components, that we do not seem to need, leads either to unphysical constraints in the theory if they are not properly taken into account – as with the additional components of Einstein's equations discussed in Section 3, or they unnecessarily restrict the space of a symmetry in the theory as with the components of ϵ^μ .

In fact, by inspecting Eqs. (5.5)-(5.6), we may easily see that an exact matching with the transformations (5.12) can still be achieved, even for a non-trivial ϵ^i , provided that the following relations hold

$$\partial_0 \epsilon_i = \partial_i \epsilon^i = 0. \quad (5.13)$$

⁶ Note that, in the case of a general coordinate transformation, we should have taken into account how the derivatives ∂_μ change as well, i.e. $\partial_\mu \rightarrow \partial'_\mu = \partial_\mu + \epsilon^\rho{}_{,\mu} \partial_\rho$. However, the latter term when acting on the perturbations $\tilde{h}_{\mu\nu}$ is of quadratic order and thus, in the linear approximation, it is ignored.

In that case, the scalar function Λ is given by the relation $\Lambda = -c^2\epsilon_0/4$, where ϵ_0 is arbitrary. The gauge invariance of the fields is valid as expected, and this holds even for non-trivial spatial components of the vector ϵ^μ , that are merely static and depending on the space coordinates according to the constraint $\partial_i\epsilon^i = 0$.

However, there is an even broader class of transformations for the potentials that leave the GEM fields (5.9) and (5.11) invariant. As their expressions show, we merely need to satisfy the following constraints

$$\partial_0^2\epsilon_i + \partial_i\partial_k\epsilon^k = 0, \quad (5.14)$$

$$\partial_0\partial_i\epsilon_j = 0. \quad (5.15)$$

To the above, we will add an additional constraint that follows from the demand that the transverse gauge condition - as it helps to restore the form of the Maxwell's equations - holds in all coordinate systems, i.e. $\partial_\nu\tilde{h}^{\mu\nu} = \partial_\nu\tilde{h}'^{\mu\nu} = 0$; this has been broadly assumed in the literature including the traditional case, and leads to the relation

$$\partial^2\epsilon^\mu = (\partial_0^2 - \partial_i^2)\epsilon^\mu = 0. \quad (5.16)$$

The temporal component ϵ^0 satisfies only Eq. (5.16), and thus will have the general form $\epsilon^0 = \epsilon^0(\mathbf{x}-ct)$. On the other hand, the spatial components ϵ^i need to satisfy all three constraints (5.14)-(5.16). Equation (5.15) demands that the time and space dependence should be separated, namely $\epsilon^i = f_i(\mathbf{x}) + g_i(t)$, for $i = 1, 2, 3$. Then, the spatial components of Eq. (5.16) dictate that this time and space dependence should be at most quadratic. In any case, we conclude that the space of the gauge symmetry of the theory can be significantly enlarged if we avoid the over-restrictive assumption of vanishing ϵ^i .

The above analysis can be repeated for the alternative ansatz of the metric perturbations (3.16). Employing Eq. (5.4), the transformation rules for Φ and \mathbf{A} are found to be given again by Eqs. (5.5)-(5.6) under the redefinitions $\Phi \rightarrow \Phi/4$ and $\mathbf{A} \rightarrow \mathbf{A}/2$. However, in the next step we encounter a problem: considering the transformation rule for the trace \tilde{h} , we obtain two different results, one if we use its transformation rule $\tilde{h}' = \tilde{h} - 2\epsilon^\rho{}_{,\rho}$ and one by employing its explicit form in terms of Φ , Eq. (3.17), and use the transformation rule for Φ itself. This is a sign that the set of transformation rules does not “close” as it is. The obstacle is easily overcome by adding to the \tilde{h}_{ij} component a term proportional to a scalar potential λ and suppressed by $1/c^4$. Then, the \tilde{h}_{ij} component reads

$$\tilde{h}_{ij} = \left(-\frac{\Phi}{c^2} + \frac{\lambda}{c^4}\right)\eta_{ij} + \frac{1}{c^4}d_{ij}. \quad (5.17)$$

Such a term was also part of the generalised traditional ansatz (3.2), and was found to have an insignificant role in our analysis; the same will hold here with its presence causing no modification to the conclusions of the previous sections regarding the alternative ansatz. However, its presence will help the set of transformation rules to “close”: indeed, assuming the form (5.17) for the

\tilde{h}_{ij} component and combining it with the transformation rules for the trace \tilde{h} and $\tilde{\Phi}$, we arrive with no inconsistencies at the complementary transformation rule

$$\lambda \rightarrow \lambda' = \lambda - \frac{4c^4}{3} \partial_i \epsilon^i, \quad (5.18)$$

while the transformation rule for d_{ij} is given again by Eq. (5.8) under the change $d_{ij} \rightarrow d_{ij}/2$.

The corresponding changes to the GEM fields \mathbf{E} and \mathbf{B} are given again by Eqs. (5.9) and (5.11), respectively, with the only difference being the rescalings $\mathbf{E} \rightarrow \mathbf{E}/4$ and $\mathbf{B} \rightarrow \mathbf{B}/4$. As a result, the most general configuration for the arbitrary vector ϵ^μ that ensures the gauge invariance of the GEM fields are still given by the set of Eqs. (5.14)-(5.16), and the same results hold. Once again, there is no need to assume a vanishing ϵ^i and the space of the symmetry is enlarged compared to previous treatments ⁷.

The inconsistency that has arisen in applying the transformation rule (5.4) to the case of the alternative ansatz is, we believe, a generic one that appears when an over-simplifying assumption is made for the form of the metric perturbations. Although the problem for the alternative ansatz was fixed quite easily, a more serious consistency problem will be present for ansatzes that are “built” to be very simple. For example, one may easily see that the traditional ansatz (2.5), where the \tilde{h}_{ij} components were altogether ignored, faces a similar problem with different transformation rules for $\tilde{\Phi}$ following from the general rule (5.4) and the trace transformation rule (5.3); unless one restores the spatial components of the metric perturbations – an approach that we have consistently followed in this work for a number of additional reasons – the problem cannot be overcome.

6 Discussion and Conclusions

The striking similarity between the gravitational and electromagnetic forces at classical level, that was found to hold also in the context of the General Theory of Relativity, has attracted a lot of attention over a century-long period. The framework of Gravito-electromagnetism, the theory that describes the dynamics of the gravitational field in terms of quantities met in electromagnetism, has also been the area of an intense activity over the years and, provided a new perspective on the description and understanding of the gravitational field.

The modern mathematical framework is that of the General Theory of Relativity, and more specifically, the perturbed Einstein’s field equations at linear approximation. The gravitational perturbations $h_{\mu\nu}$ involve directly the

⁷ Note that, according to Eq. (5.18), the absence of λ would be admissible, and no inconsistencies with the closure of the transformation rules would arise, if the condition $\partial_i \epsilon^i = 0$ was demanded. This condition is one of the two (5.13) assumed when the perfect analogy with the EM is sought for. However, as we saw earlier this leads to a more restrictive form of the vector ϵ^μ than it is necessary. For this reason, above, we have followed instead the option of the introduction of the λ term that, while not affecting any of our results, helps to keep the space of the symmetry as large as possible.

GEM potentials Φ and \mathbf{A} , and the field equations then reduce to a set of Maxwell-like equations for the GEM fields \mathbf{E} and \mathbf{B} defined in terms of the potentials in the usual electro-magnetic way. In this, and in a previous work of ours [15], we have shown that the choice of the form of the gravitational perturbations is of paramount importance for the successful restoration of Maxwell's equations and the validity of the analogy between gravity and EM. In section 2, after reviewing the most basic assumptions and equations of GEM, we discussed the weak points of the analysis based on the so-called traditional ansatz (2.5) for the gravitational perturbations [12]: (i) although the Maxwell's equations are restored, this is realised only for a static vector potential \mathbf{A} , (ii) the form of the Lorentz equation is reproduced but potentially important new terms, not present in EM, can be avoided only under the assumption of a static scalar potential Φ , (iii) the additional components of the field equations and gauge condition, that were ignored, can impose unphysical or over-restrictive constraints to the fields or matter distribution of the theory.

In the light of the above results, in Section 3, we introduced two novel forms of metric perturbations. The first one was a generalisation of the traditional ansatz, Eq. (3.2), in the sense that allowed small but non-vanishing spatial components of the metric perturbations $\tilde{h}_{\mu\nu}$. The \tilde{h}_{ij} components are indeed not involved in the derivation of Maxwell's equations, and one may easily discard them as irrelevant. However, due to the tensorial structure of the theory the absence or presence of these quantities affect other important equations of the theory. Thus, in Section 3.1, we explicitly demonstrated that, in the absence of \tilde{h}_{ij} , the additional component of the transverse gauge condition *demand*s the staticity of the vector potential \mathbf{A} . On the other hand, in its presence, important terms, involving time-derivatives of \mathbf{A} that were previously missing, are now restored, and the set of Maxwell's equations are exactly reproduced. Moreover, all additional constraints were rendered harmless, and simply connected the form of the \tilde{h}_{ij} components to the distribution of matter in the theory.

The expression of the Lorentz equation, though, still suffered from the presence of additional terms that may not be easily ignored. In search of a new ansatz that could perhaps achieve both objectives at once, i.e. restore Maxwell's equations and the Lorentz equation, in Section 3.2 we proposed the so-called alternative ansatz (3.16). This comprised a variant of a similarly named ansatz for the metric perturbations proposed in our previous work [15] its core idea being the introduction of the scalar potential Φ into the spatial components \tilde{h}_{ij} , too, of the metric perturbations; it was then showed that a cancellation of terms coming from the \tilde{h}_{00} and \tilde{h}_{ij} components led to an expression for the Lorentz equation that was identical to that of EM and contained no additional terms. In the context of the present work, we extended further our previous ansatz, in a way similar to that adopted in Section 3.1, in order to avoid again the unphysical constraints in the theory. Indeed, employing the ansatz (3.16), all potentially harmful terms in the expression of the Lorentz force were eliminated, and the set of Maxwell's equations was restored with no conditions imposed on the field configurations – the only point of disagree-

ment, a superfluous coefficient of 4, could be absorbed into the redefinition of the matter/charge density in the context of GEM.

Pursuing further the analogy between gravity and EM, in Section 4 we searched for scalar invariant quantities defined in terms of gravitational tensors in the context of the linear approximation employed in GEM. In Section 4.1, we defined a novel 3rd-rank tensor $F_{\alpha\mu\nu}$ – an analogue of the electromagnetic field strength tensor $F_{\mu\nu}$ – and the scalar combination $F^{\alpha\mu\nu}F_{\alpha\mu\nu}$ was computed. For both metric ansatzes (3.2) and (3.16), it was found that, although this scalar combination resembled the one of $F^{\mu\nu}F_{\mu\nu}$ in EM, the presence of additional terms of the same order as \mathbf{E}^2 and \mathbf{B}^2 rendered this combination not satisfactory. In Section 4.2, we proceeded to define two novel 3rd-rank tensors $Q^{\alpha\mu\nu}$ and $H^{\alpha\mu\nu}$ motivated by the fact that the combination $-Q^{\alpha\mu\nu}H_{\alpha\mu\nu}/4$ exactly matches the gravitational Lagrangian in the linear approximation. Here, the results were very encouraging: the aforementioned scalar quantity was indeed proportional to a linear combination of \mathbf{E}^2 and \mathbf{B}^2 , as expected, with no additional terms of the same order being present. Enforced by these results, we then demonstrated that a second scalar invariant quantity of the form $\tilde{Q}^{\alpha\mu\nu}H_{\alpha\mu\nu}$, where $\tilde{Q}^{\alpha\mu\nu}$ is the dual of $Q^{\alpha\mu\nu}$ – an analogue again of the $\tilde{F}^{\mu\nu}F_{\mu\nu}$ quantity in EM – could be defined. For both metric ansatzes, this second invariant quantity was found to be indeed proportional to the internal product $\mathbf{E} \cdot \mathbf{B}$, with no additional corrections of the same order.

In the last section, section 5, we investigated in detail the conditions under which the gauge invariance of the linearised gravitational theory leads to a gauge invariance of the GEM fields \mathbf{E} and \mathbf{B} . For a small displacement of the coordinates, parametrised by an arbitrary vector ϵ^μ , the transformation rule for the metric perturbations $h_{\mu\nu}$ was found, and from that the transformation relations for the GEM potentials and fields were derived. As has been shown in the literature [12], the latter transformation relations reduce to the usual gauge transformations of EM under the assumption that the spatial components of ϵ^μ vanish. Here, using our two novel ansatzes for the metric perturbations (3.2) and (3.16), that were shown to lead to more satisfactory results than the traditional ansatz (2.5) regarding the restoration of Maxwell's and Lorentz equations, we have demonstrated that a gauge invariance of the GEM fields \mathbf{E} and \mathbf{B} holds for a much more general form of the arbitrary vector ϵ^μ . The conditions that the temporal and spatial components of ϵ^μ need to satisfy were derived, and their general configuration was determined. In this way, an over-restrictive assumption regarding the form of the arbitrary vector ϵ^μ was avoided, and the space of the symmetry present in the theory was enlarged compared to previous studies.

In the previous work of ours [15], we had also posed the question whether a tensorial theory based on the formalism of General Relativity could exactly re-produce the theory of Electromagnetism. The starting point of that analysis was again the linearised Einstein's equations with the constant k appearing on the right-hand-side being initially arbitrary instead of taking the usual value of $8\pi G/c^4$. Had we repeated that analysis by employing our two novel ansatzes for the metric perturbations, a number of satisfactory results would

have emerged: (i) for the generalised traditional ansatz (3.2), all Maxwell's EM equations would have been exactly reproduced under the assumption that $k = -2\pi/c^4$; however, the additional terms in the expression of the Lorentz force would now have dire phenomenological consequences, (ii) for the alternative ansatz (3.16), though, these terms would be altogether absent, with the expression of the Lorentz force exactly matching the one in EM and the only additional corrections being of order $\mathcal{O}(1/c^4)$ and thus negligible both in the non-relativistic and relativistic limit; in addition, the set of Maxwell's equations would have been again reproduced, now under the assumption that $k = -8\pi/c^4$, and for the exact same definitions of the fields \mathbf{E} and \mathbf{B} in terms of the potentials Φ and \mathbf{A} as in true EM. We thus conclude that the use of the alternative ansatz has finally provided the proper form of metric perturbations that, in conjunction with Einstein's tensorial equations, has reproduced the complete mathematical framework of the theory of electromagnetism.

Naturally, the above suggestion does not aim at replacing the theory of electromagnetism by a tensorial theory; it rather aims at investigating how far the analogy between gravity and electromagnetism extends, given the persistent difficulty in finding a common mathematical framework to unify gravity and gauge forces. Moreover, as it has been the objective of the theory of GEM itself, this analogy may open new directions and lead to the discovery of new phenomena in gravitational physics guided by phenomena already observed in EM. The two novel ansatzes that we have proposed here have come a long way in overcoming previous inconsistencies of GEM and in achieving the best analogy between gravity and EM in the literature so far. Our results have also improved a previous analysis of ours in the sense that the set of perturbed equations derived in this work are free of any unphysical constraints and valid both far away and close to the sources of the gravitational field; this, may prove to be of significant importance in future applications of our formalism to phenomena in gravitational physics, especially in the light of the recent discovery of gravitational waves whose study is based on the linear approximation method.

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A Components of the Christoffel symbols

Here, we present the components of the Christoffel symbols for both metric perturbations ansatzes that are necessary for the derivation of the Lorentz equation in each case.

Employing the expression of the Christoffel symbols (3.9) in the linear approximation and of the initial perturbations $h_{\mu\nu}$ from the line-element (3.12) for the case of the generalised traditional ansatz, we find the following components

$$\begin{aligned} \Gamma_{00}^i &= \frac{1}{c^2} \partial_i \Phi - \frac{3}{2c^4} \partial_i \lambda + \frac{2}{c^3} \partial_t A^i, \\ \Gamma_{0j}^i &= \frac{1}{c^2} F_{ij} - \frac{1}{c^3} \delta_j^i \left(\partial_t \Phi + \frac{1}{2c^2} \partial_t \lambda \right) + \frac{1}{c^5} \partial_t d_j^i, \end{aligned} \quad (\text{A.1})$$

$$\begin{aligned} \Gamma_{kj}^i &= -\frac{1}{c^2} \left(\delta_j^i \partial_k \Phi + \delta_k^i \partial_j \Phi - \delta_{kj} \partial_i \Phi \right) \\ &\quad - \frac{1}{2c^4} \left(\delta_j^i \partial_k \lambda + \delta_k^i \partial_j \lambda - \delta_{kj} \partial_i \lambda \right) + \frac{1}{c^4} \left(\partial_k d_j^i + \partial_j d_k^i - \partial^i d_{kj} \right). \end{aligned}$$

We note that, in the second of the above equations, we have used the definition $F_{ij} \equiv \partial_i A_j - \partial_j A_i$.

In the case of the alternative ansatz for the metric perturbations, the original perturbations $h_{\mu\nu}$ may be deduced from the line-element (3.18). Then, the Christoffel symbols are found to be

$$\begin{aligned} \Gamma_{00}^i &= \frac{(1+3\gamma)}{4c^2} \partial_i \Phi + \frac{1}{c^3} \partial_t A^i, & \Gamma_{0j}^i &= \frac{1}{2c^2} F_{ij} - \frac{(1-3\gamma)}{4c^3} \delta_j^i \partial_t \Phi + \frac{1}{2c} \partial_t \tilde{h}_{ij}^i, \\ \Gamma_{kj}^i &= -\frac{(1-3\gamma)}{4c^2} \left(\delta_j^i \partial_k \Phi + \delta_k^i \partial_j \Phi - \delta_{kj} \partial_i \Phi \right) + \frac{1}{2} \left(\partial_k \tilde{h}_{ij}^i + \partial_j \tilde{h}_{ik}^i - \partial^i \tilde{h}_{kj} \right). \end{aligned} \quad (\text{A.2})$$

B Components of the novel gravitational tensors

By using the components of the perturbations $\tilde{h}_{\mu\nu}$, as these appear in the generalised traditional ansatz (3.2), we find the following explicit forms for the components of the new tensor $F_{\alpha\mu\nu}$, defined in Eq. (4.6):

$$\begin{aligned} F_{000} &= -\frac{2}{c^2} \partial_i A^i, & F_{i00} &= \frac{4}{c^2} E^i - \frac{2}{c^4} \left(\partial_i \lambda + \partial^j d_{ij} \right), \\ F_{00i} &= \frac{4}{c^2} \partial_i \Phi, & F_{ij0} &= -\frac{2}{c^2} F_{ij} + \frac{2}{c^5} \partial_t (\lambda \eta_{ij} + d_{ij}), \\ F_{0ij} &= \frac{2}{c^2} \left[(\partial_i A_j + \partial_j A_i) - \eta_{ij} \left(\frac{2}{c} \partial_t \Phi + \partial_k A^k \right) \right] - \frac{2}{c^5} \partial_t (\lambda \eta_{ij} + d_{ij}), \\ F_{ijk} &= \frac{2}{c^4} \left[\partial_j (\lambda \eta_{ik} + d_{ik}) - \partial_k (\lambda \eta_{ij} + d_{ij}) - \partial_i (\lambda \eta_{jk} + d_{jk}) \right] \\ &\quad - \frac{2}{c^3} \eta_{jk} \left[\partial_t A_i + \frac{1}{c} \partial^l (\lambda \eta_{il} + d_{il}) \right], \end{aligned} \quad (\text{B.1})$$

where the GEM fields are defined as

$$E^i \equiv -\frac{1}{c} \partial_t \left(\frac{A^i}{2} \right) - \partial_i \Phi, \quad F_{ij} \equiv \partial_i A_j - \partial_j A_i = -2\epsilon_{ijk} B_k. \quad (\text{B.2})$$

Similarly, employing the components of $\tilde{h}_{\mu\nu}$ of the alternative ansatz (3.16), we find the results

$$\begin{aligned} F_{000} &= -\frac{1}{c^2} \partial_i A^i, & F_{i00} &= \frac{1}{c^3} \partial_t A_i - \frac{1}{c^4} \partial^j d_{ij}, & F_{00i} &= \frac{1}{c^2} \partial_i \Phi, \\ F_{ij0} &= -\frac{1}{c^2} F_{ij} - \frac{1}{c^3} \partial_t \Phi \eta_{ij} + \frac{1}{c^5} \partial_t d_{ij}, \\ F_{0ij} &= \frac{1}{c^2} \left[(\partial_i A_j + \partial_j A_i) - \eta_{ij} \partial_k A^k \right] - \frac{1}{c^5} \partial_t d_{ij}, \\ F_{ijk} &= -\frac{1}{c^2} \left(\eta_{ij} \partial_k \Phi + \eta_{ik} \partial_j \Phi - 2\eta_{jk} \partial_i \Phi \right) - \frac{1}{c^3} \eta_{jk} \partial_t A_i \\ &\quad + \frac{1}{c^4} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk}) - \frac{1}{c^4} \eta_{jk} \partial^l d_{il}. \end{aligned} \quad (\text{B.3})$$

We now move to the components of the gravitational tensors $Q_{\alpha\mu\nu}$ and $H_{\alpha\mu\nu}$ defined in Eqs. (4.15). For their calculation, we first need the components of the fully-covariant Christoffel symbols (4.16). For the generalised ansatz (3.2), these are listed below

$$\begin{aligned}
2\Gamma_{000} &= \frac{2}{c^3} \partial_t \Phi, & 2\Gamma_{00i} &= \frac{2}{c^2} \partial_i \Phi, & 2\Gamma_{i00} &= \frac{4}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, \\
2\Gamma_{0ij} &= \frac{2}{c^2} (\partial_i A_j + \partial_j A_i) + \frac{2}{c^3} \partial_t \Phi - \frac{2}{c^5} \partial_t d_{ij}, \\
2\Gamma_{i0j} &= -\frac{2}{c^2} F_{ij} - \eta_{ij} \frac{2}{c^3} \partial_t \Phi + \frac{2}{c^5} \partial_t d_{ij} \\
2\Gamma_{ijk} &= -\frac{2}{c^2} (\eta_{ij} \partial_k \Phi + \eta_{ik} \partial_j \Phi - \eta_{jk} \partial_i \Phi) + \frac{2}{c^4} (\partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}).
\end{aligned} \tag{B.4}$$

For simplicity, in the above we have set $\lambda = 0$. Then, along with Eq. (3.11) and the relation $h = -\tilde{h}$, we find the following components of the $Q_{\alpha\mu\nu}$

$$\begin{aligned}
Q_{000} &= \frac{6}{c^3} \partial_t \Phi, & Q_{i00} &= \frac{4}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, & Q_{00i} &= \frac{2}{c^2} \partial_i \Phi, \\
Q_{0i0} &= \frac{6}{c^2} \partial_i \Phi, & Q_{ij0} &= -\frac{2}{c^2} F_{ij} - \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{2}{c^5} \partial_t d_{ij}, \\
Q_{0ij} &= \frac{2}{c^2} (\partial_i A_j + \partial_j A_i) + \frac{2}{c^3} \eta_{ij} \partial_t \Phi - \frac{2}{c^5} \partial_t d_{ij}, \\
Q_{i0j} &= -\frac{2}{c^2} F_{ij} + \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{2}{c^5} \partial_t d_{ij}, \\
Q_{ijk} &= -\frac{2}{c^2} (\eta_{ij} \partial_k \Phi - \eta_{ik} \partial_j \Phi - \eta_{jk} \partial_i \Phi) + \frac{2}{c^4} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk}).
\end{aligned} \tag{B.5}$$

and $H_{\alpha\mu\nu}$ tensors

$$\begin{aligned}
H_{000} &= \frac{6}{c^3} \partial_t \Phi, & H_{i00} &= \frac{2}{c^2} \partial_i \Phi, & H_{00i} &= \frac{6}{c^2} \partial_i \Phi, \\
H_{0i0} &= \frac{4}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, & H_{ij0} &= \frac{2}{c^2} F_{ij} + \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{2}{c^5} \partial_t d_{ij}, \\
H_{0ij} &= -\frac{2}{c^2} F_{ij} - \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{2}{c^5} \partial_t d_{ij}, \\
H_{i0j} &= \frac{2}{c^2} (\partial_i A_j + \partial_j A_i) + \frac{2}{c^3} \eta_{ij} \partial_t \Phi - \frac{2}{c^5} \partial_t d_{ij} \\
H_{ijk} &= \frac{2}{c^2} (\eta_{ij} \partial_k \Phi + \eta_{ik} \partial_j \Phi - \eta_{jk} \partial_i \Phi) + \frac{2}{c^4} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk}).
\end{aligned} \tag{B.6}$$

We follow a similar analysis for the case of the alternative ansatz (3.16). Here, the components of the fully-covariant Christoffel symbols have the form

$$\begin{aligned}
2\Gamma_{000} &= \frac{2}{c^3} \partial_t \Phi, & 2\Gamma_{00i} &= \frac{2}{c^2} \partial_i \Phi, & 2\Gamma_{i00} &= \frac{2}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, \\
2\Gamma_{0ij} &= \frac{1}{c^2} (\partial_i A_j + \partial_j A_i) - \frac{1}{c^5} \partial_t d_{ij}, & 2\Gamma_{i0j} &= -\frac{1}{c^2} F_{ij} + \frac{1}{c^5} \partial_t d_{ij} \\
2\Gamma_{ijk} &= \frac{1}{c^4} (\partial_k d_{ij} + \partial_j d_{ik} - \partial_i d_{jk}).
\end{aligned} \tag{B.7}$$

The trace of the original perturbations is given again by the relation $h = -\tilde{h}$ and the use of Eq. (3.17) (with $\gamma = 1$). By combining the above, we find the following components for the

$Q_{\alpha\mu\nu}$

$$\begin{aligned}
Q_{000} &= 0 = Q_{0i0}, & Q_{i00} &= \frac{2}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, & Q_{00i} &= \frac{2}{c^2} \partial_i \Phi, \\
Q_{ij0} &= -\frac{1}{c^2} F_{ij} + \frac{1}{c^5} \partial_t d_{ij} & Q_{0ij} &= \frac{1}{c^2} (\partial_i A_j + \partial_j A_i) - \frac{1}{c^5} \partial_t d_{ij}, & \\
Q_{i0j} &= -\frac{1}{c^2} F_{ij} - \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{1}{c^5} \partial_t d_{ij}, \\
Q_{ijk} &= -\frac{2}{c^2} \eta_{ik} \partial_j \Phi + \frac{1}{c^4} (\partial_j d_{ik} + \partial_k d_{ij} - \partial_i d_{jk}).
\end{aligned} \tag{B.8}$$

and $H_{\alpha\mu\nu}$ tensors

$$\begin{aligned}
H_{000} &= 0 = H_{00i}, & H_{i00} &= \frac{2}{c^2} \partial_i \Phi, & H_{0i0} &= \frac{2}{c^3} \partial_t A_i - \frac{2}{c^2} \partial_i \Phi, \\
H_{ij0} &= \frac{1}{c^2} F_{ij} - \frac{2}{c^3} \partial_t \Phi \eta_{ij} + \frac{1}{c^5} \partial_t d_{ij}, & H_{0ij} &= -\frac{1}{c^2} F_{ij} + \frac{1}{c^5} \partial_t d_{ij}, \\
H_{i0j} &= \frac{1}{c^2} (\partial_i A_j + \partial_j A_i) - \frac{1}{c^5} \partial_t d_{ij} \\
H_{ijk} &= -\frac{2}{c^2} \eta_{ij} \partial_k \Phi + \frac{1}{c^4} (\partial_k d_{ij} + \partial_i d_{jk} - \partial_j d_{ik}).
\end{aligned} \tag{B.9}$$

For the construction of the dual invariant quantity $A_2 = \tilde{Q}H = \tilde{H}Q = -\tilde{Q}Q = -\tilde{H}H$, we merely need the components of the fully-covariant Christoffel symbols for the two metric ansatzes, listed in Eqs. (B.4) and (B.7), respectively, and the general expression (4.26).

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